Université Paris VII and I

Master 2 M0 2018 – 2019

Analyse des séries financières

Examen final, Février 2019

3h00, sans aucun document

1. Let $(\varepsilon_t)_{t\in\mathbf{Z}}$ a sequence of centered independent and identically distributed random variables such as $\mathbb{E}(\varepsilon_0^2)=1$. If it exists we consider a sequence $(X_t)_{t\in\mathbf{Z}}$ such as:

$$X_t=\varepsilon_t\,\sigma_t \quad \text{with} \quad \sigma_t=a_0+a_1\,|X_{t-1}|+b_1\,\sigma_{t-1} \quad \text{for any } t\in\mathbf{Z} \tag{1}$$

where $(a_0,a_1,b_1)\in[0,\infty)^3$ are unknown parameters with $b_1>0$.

(a) Prove that (ε_t) is a stationary time series.

(b) Denote $\mathbf{R}^\infty$ the space of sequences of real numbers with finite number of non-zero real numbers. Consider $F:\mathbf{R}^\infty\to\mathbf{R}$ be a measurable function on $\mathbf{R}^\infty$. Assume also that F is Lipchitzian on $\mathbf{R}^\infty$: there exists a sequence $(\ell_i)_{i\in\mathbf{N}^*}$ of real non-negative numbers such that for any $x=(x_i)_i,\ y=(y_i)_i\in\mathbf{R}^\infty$,

$$\big|F(x)-F(y)\big|\le\sum_{i=1}^\infty \ell_i\,|x_i-y_i|, \quad \text{where} \quad \sum_{i=1}^\infty \ell_i<\infty.$$

For $t\in\mathbf{Z}$, prove that $(Y_t^{(n)})_n$ where $Y_t^{(n)}=F\big(\varepsilon_t,\varepsilon_{t-1},\dots,\varepsilon_{t-n},0,0,\dots\big)$ is a Cauchy sequence on $\mathbb{L}^2$. Deduce that $(Y_t)_{t\in\mathbf{Z}}$, where $Y_t=F\big((\varepsilon_{t-k})_{k\in\mathbf{N}}\big)$ for $t\in\mathbf{Z}$, is a stationary second order process.

(c) Let $\alpha(\cdot)$ be the function such as $\alpha(x)=a_1|x|+b_1$. Show that $\mathbb{E}\big[\big|\log\big(\alpha(\varepsilon_0)\big)\big|\big]<\infty$ using Jensen Inequality. Deduce also that for any $t\in\mathbf{Z}$:

$$\Big(\prod_{i=1}^n\alpha(\varepsilon_{t-i})\Big)^{1/n}\xrightarrow[n\to+\infty]{a.s.} e^\gamma \quad \text{with} \quad \gamma=\mathbb{E}\big[\log\big(\alpha(\varepsilon_0)\big)\big]. \tag{2}$$

(d) Suppose $(\sigma_t)_{t\in\mathbf{Z}}$ exists in (1). Then, establish that $\sigma_t=a_0+\alpha(\varepsilon_{t-1})\,\sigma_{t-1}$ for any $t\in\mathbf{Z}$. Deduce by recurrence that for any $t\in\mathbf{Z}$ and $k\in\mathbf{N}^*$,

$$\sigma_t=\beta_t(k)+\alpha(\varepsilon_{t-1})\times\cdots\times\alpha(\varepsilon_{t-k})\sigma_{t-k} \quad \text{with} \quad \beta_t(k)=a_0\Big(1+\sum_{i=1}^{k-1}\prod_{j=1}^{i}\alpha(\varepsilon_{t-j})\Big).$$

(e) Using (2) prove that if $\gamma<0$ then for any $t\in\mathbf{Z}$, $\beta_t(n)\xrightarrow[n\to+\infty]{a.s.}\beta_t$ with β_t a positive random variable and satisfies $\beta_t=a_0+\alpha(\varepsilon_{t-1})\beta_{t-1}$. Consequently, write $X_t=F(\varepsilon_t,\varepsilon_{t-1},\dots)$ and conclude about the existence and stationarity of $(X_t)_{t\in\mathbf{Z}}$. What's happening if $a_0=0$?

(f) Assume now $\gamma<0$ and $b_1<1$. Using an iterating decomposition, show for any $t\in\mathbf{Z}$:

$$\sigma_t=\frac{a_0}{1-b_1}+a_1\sum_{j=0}^\infty b_1^j|X_{t-j-1}|.$$

Deduce $\mathbb{E}(X_t\mid X_{t-1},X_{t-2},\dots)$ and $\mathrm{var}(X_t\mid X_{t-1},X_{t-2},\dots)$. Is (X_t) a conditionaly heteroskedastic process?

(g) Assume now that $(X_1, \cdots, X_N)$ is observed and let $\theta = {}^t(a_0, a_1, b_1)$. Prove that the quasi-maximum likelihood estimator $\widehat{\theta}$ of θ is:

$$\widehat{\theta} = \operatorname*{argmin}_{\theta \in \Theta} \Big\{ \log\Big(\frac{a_0}{1-b_1}\Big) + \frac{1}{2}\Big(\frac{(1-b_1)X_1}{a_0}\Big)^2 + \sum_{i=2}^{N} \log\Big(\frac{a_0}{1-b_1} + a_1 \sum_{j=0}^{i-2} b_1^j |X_{i-j-1}|\Big) + \frac{1}{2}\Big(\frac{X_i}{\frac{a_0}{1-b_1} + a_1 \sum_{j=0}^{i-2} b_1^j |X_{i-j-1}|}\Big)^2 \Big\}$$

where Θ is a set that should be specified.

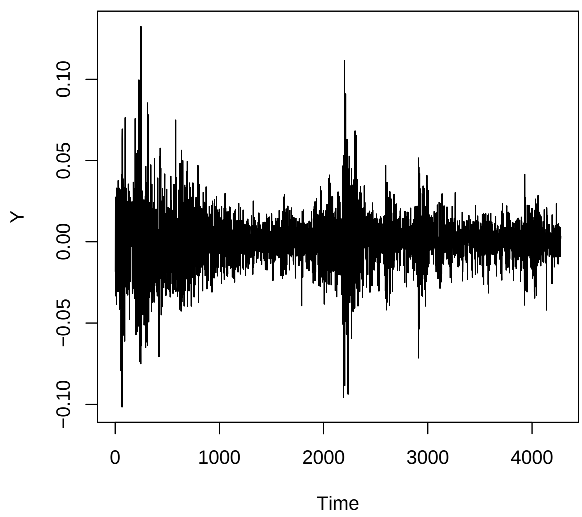
(h) Is $\widehat{\theta}$ a consistent estimator? What is its convergence rate?

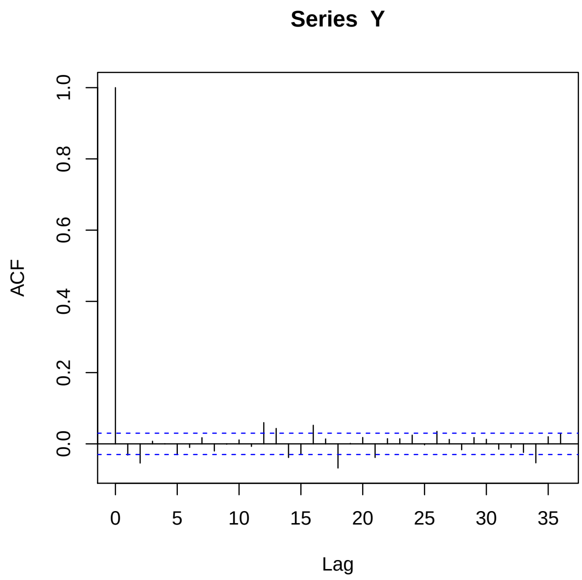
(i) Provide forecasting of X_{N+1} and X_{N+1}^2.

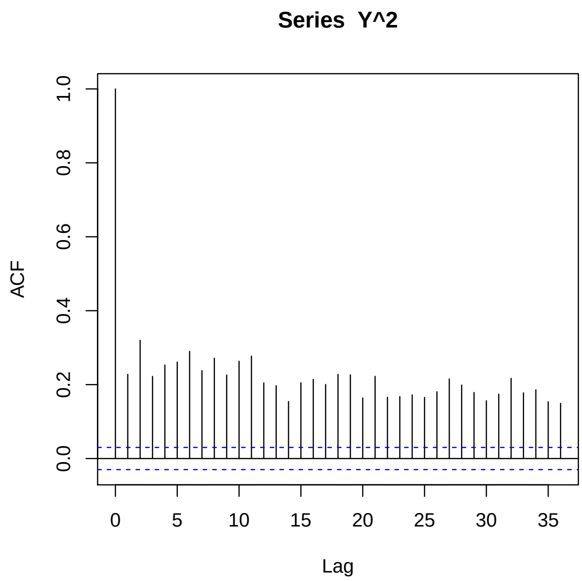
2. We study with R software the Nasdaq index from January 11st 2000 to January 11 2016.

(a) First the following commands have been executed with figures exhibited below:

```
Nas=read.csv("C:/Users/Admin/Dropbox/Enseignement/M2 MMMEF/TP/Nasdaq.csv")
X=Nas$Closing
n=length(X)
Y=log(X[2:n]/X[1:(n-1)])
ts.plot(Y)
acf(Y)
acf(Y^2)
```







Question II.1: Explain what is done. Which conclusions could you obtain from both the last commands? Are they compatible with a GARCH modelling?

(b) New commands are then executed:

```
library(fGarch)
QMLE=garchFit(~ garch(1,1), data = Y, trace = FALSE)
QMLE
```

Here there are the numerical results:

```
Call:
 garchFit(formula = ~garch(1, 1), data = Y, trace = FALSE)

Coefficient(s):
        mu       omega      alpha1       beta1
6.4040e-04  2.1146e-06  8.6732e-02  9.0314e-01

        Estimate  Std. Error  t value Pr(>|t|)
mu     6.404e-04   1.617e-04    3.961 7.46e-05 ***
omega  2.115e-06   4.052e-07    5.219 1.80e-07 ***
alpha1 8.673e-02   8.639e-03   10.039  < 2e-16 ***
```

```
beta1  9.031e-01   9.165e-03   98.544  < 2e-16 ***

                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  168.5587  0
 Shapiro-Wilk Test  R    W      0.9923722 2.300106e-14
 Ljung-Box Test     R    Q(10)  10.64455  0.3858725
 Ljung-Box Test     R    Q(15)  17.2698   0.3029932
 Ljung-Box Test     R    Q(20)  26.26701  0.1571734
 Ljung-Box Test     R^2  Q(10)  20.02718  0.02899662
 Ljung-Box Test     R^2  Q(15)  31.17497  0.008323147
 Ljung-Box Test     R^2  Q(20)  35.92566  0.01569342
 LM Arch Test       R    TR^2   20.11987  0.06485208

Information Criterion Statistics:
      AIC       BIC       SIC      HQIC
-5.886147 -5.880197 -5.886149 -5.884045
```

Question II.2: Explain what is done and explain which conclusions you deduce.

(c) Finally the following sequence of commands are executed:

```
M=matrix(0,3,4)
NAS10=garchFit(~ garch(1,0), data = Y, trace = FALSE)
M[1,1]=NAS10@fit$ics[2]
NAS11=garchFit(~ garch(1,1), data = Y, trace = FALSE)
M[1,2]=NAS11@fit$ics[2]
NAS12=garchFit(~ garch(1,2), data = Y, trace = FALSE)
M[1,3]=NAS12@fit$ics[2]
NAS13=garchFit(~ garch(1,3), data = Y, trace = FALSE)
M[1,4]=NAS13@fit$ics[2]
NAS20=garchFit(~ garch(2,0), data = Y, trace = FALSE)
M[2,1]=NAS20@fit$ics[2]
NAS21=garchFit(~ garch(2,1), data = Y, trace = FALSE)
M[2,2]=NAS21@fit$ics[2]
NAS22=garchFit(~ garch(2,2), data = Y, trace = FALSE)
..................................................
NAS33=garchFit(~ garch(3,3), data = Y, trace = FALSE)
M[3,4]=NAS33@fit$ics[2]
M
(Gop=which(M==min(M),2))
summary(NAS21)
```

The resuts are the following:

```
> M
          [,1]      [,2]      [,3]      [,4]
[1,] -5.501540 -5.880197 -5.878215 -5.876146
[2,] -5.665021 -5.881333 -5.879577 -5.877575
[3,] -5.717491 -5.879332 -5.877748 -5.875797
> (Gop=which(M==min(M),2))
     row col
[1,]   2   2
```

Question II.3: What is done here and what are your conclusions?