*Université Paris VII - Paris I*

**Master 2 M0** 2017 – 2018

# Analyse des séries financières

Final exam, February 2018

*3h00, without any documents.*

In all the sequel, $(\xi_t)_{t\in\mathbf{Z}}$ is a sequence of centered independent and identically distributed random variables following a symmetric distribution (*i.e.*, the distributions of $\xi_0$ and $-\xi_0$ are the same) continuous with respect to Lebesgue measure and such as $\mathbb{E}(\xi_0^2)=1$.

1. **Preliminaries:**

   (a) Let $(X_t)_{t\in\mathbf{Z}}$ be a stationary time series. Prove that $(X_t^2)_{t\in\mathbf{Z}}$ is a stationary time series.

   (b) Assume that $(Y_t^2)_{t\in\mathbf{Z}}$ is a stationary time series. Prove that $(Y_t)_{t\in\mathbf{Z}}$ is not necessary a stationary time series.

   (c) Let $(u_t)_{t\in\mathbf{Z}}$ be a stationary time series, independent to $(\xi_t)_{t\in\mathbf{Z}}$. Prove that $(\xi_t\, u_t)_{t\in\mathbf{Z}}$ is a stationary time series.

   (d) For a random variable $Z$, define $\mathrm{sign}(Z)=\mathbb{I}_{Z>0}-\mathbb{I}_{Z<0}$. Prove that $(\mathrm{sign}(\xi_t))_{t\in\mathbf{Z}}$ is a white noise, independent of $(|\xi_t|)_{t\in\mathbf{Z}}$. Let $(Y_t)_{t\in\mathbf{Z}}$ be a time series defined by $Y_t=\xi_t\, G\big((|Y_{t-i}|)_{i\in\mathbf{N}^*}\big)$ for any $t\in\mathbf{Z}$ where $G:\mathbf{R}^{\mathbf{N}}\to(0,\infty)$ is a fixed function. Assume that $(Y_t^2)_{t\in\mathbf{Z}}$ is a causal (with respect to $((\xi_s)_{s\le t})_{t\in\mathbf{Z}}$) stationary process. Prove that $(Y_t)_{t\in\mathbf{Z}}$ is also a causal stationary time series.

*Proof.* (a) Clear since $g(X_{t_1},\dots,X_{t_k})\overset{\mathcal{D}}{\sim} g(X_{t_1+c},\dots,X_{t_k+c})$ when $g:\mathbf{R}^k\to\mathbf{R}^k$ is a Borelian function.

(b) If $X_t=(-1)^t$ for any $t\in\mathbf{Z}$, then $(X_t)$ is not a stationary process, but $(X_t^2)$ is a stationary process.

(c) We have $(u_{t_1},\dots,u_{t_k})\overset{\mathcal{D}}{\sim}(u_{t_1+c},\dots,u_{t_k+c})$ and $(\xi_{t_1},\dots,\xi_{t_k})\overset{\mathcal{D}}{\sim}(\xi_{t_1+c},\dots,\xi_{t_k+c})$. Since $(u_{t_1},\dots,u_{t_k})$ and $(\xi_{t_1},\dots,\xi_{t_k})$ are independent, then $(u_{t_1},\dots,u_{t_k},\xi_{t_1},\dots,\xi_{t_k})\overset{\mathcal{D}}{\sim}(u_{t_1+c},\dots,u_{t_k+c},\xi_{t_1+c},\dots,\xi_{t_k+c})$. Apply now $h:\mathbf{R}^{2k}\to\mathbf{R}^k$ such as $h(x_1,\dots,x_{2k})=(x_1x_{k+1},\dots,x_kx_{2k})$ at both sides of the previous equality.

(d) It is clear that $(\mathrm{sign}(\xi_t))_{t\in\mathbf{Z}}$ is a sequence of iidrv since $(\xi_t)_{t\in\mathbf{Z}}$ is a sequence of iidrv and $\mathbb{E}(\mathrm{sign}(\xi_t))=0$ (symmetry) for any $t$ and $\mathrm{var}(\mathrm{sign}(\xi_t))=Cte>0$ since the law of $\xi_0$ is not the Dirac measure in 0. Moreover, for any $t$, $\mathrm{sign}(\xi_t)$ is independent to $(|\xi_s|)_{s\neq t}$ since $(\xi_t)_{t\in\mathbf{Z}}$ is a sequence of iidrv. The only thing to prove is $|\xi_0|$ independent to $\mathrm{sign}(\xi_0)$. For $x\ge 0$, $\mathbb{P}(|\xi_0|\le x\mid \mathrm{sign}(\xi_0)=1)=\mathbb{P}(0\le\xi_0\le x\mid\mathrm{sign}(\xi_0)=1)=\mathbb{P}(0\le\xi_0\le x)/P(\mathrm{sign}(\xi_0)=1)=2\,\mathbb{P}(0\le\xi_0\le x)=2\,\mathbb{P}(-x\le\xi_0\le 0)=\mathbb{P}(|\xi_0|\le x\mid\mathrm{sign}(\xi_0)=1)$ from symmetry of the distribution of $\xi_0$ (the case $\xi_0=0$ has not to be considered wince it is with null probability).
If $(Y_t^2)_{t\in\mathbf{Z}}$ is a causal stationary process, with the same trick than in (a), we deduce $((|Y_t|)_{t\in\mathbf{Z}}$ is a stationary process. But $|Y_t|=|\xi_t|\,G^{1/2}\big((|Y_{t-i}|)_{i\in\mathbf{N}^*}\big)$, and since it is also causal and since $(\xi_t)_{t\in\mathbf{Z}}$ is a sequence of iidrv, $(\mathrm{sign}(\xi_t))$ is independent to $(|Y_t|)$. Finally, since $Y_t=\mathrm{sign}(\xi_t)\,|Y_t|$, and using (c), we deduce that $(Y_t)$ is also a causal stationary process.

$\square$

2. **Main theoretical part:** If it exists, we consider a sequence $(X_t)_{t\in\mathbf{Z}}$ such as:

$$X_t=\alpha\,X_{t-1}+\varepsilon_t \quad\text{with}\quad \varepsilon_t=\xi_t\sqrt{a_0+a_1X_{t-1}^2}\quad\text{for any } t\in\mathbf{Z} \tag{1}$$

   where $(\alpha,a_0,a_1)\in]=\mathbf{R}\times(0,\infty)\times[0,\infty)$ are unknown parameters.

   (a) In this question, we assume $\alpha=a_1=0$. Which kind of process is $(X_t)_{t\in\mathbf{Z}}$ and provide a condition of the existence of a stationary causal second order solution. Compute $\mathbb{E}(X_0)$ and $r_X(k)=\mathrm{cov}(X_0,X_k)$ for $k\in\mathbf{N}$.

   (b) In this question, we assume $a_1=0$ and $\alpha\neq 0$. Which kind of process is $(X_t)_{t\in\mathbf{Z}}$ and provide a condition of the existence of a stationary causal second order solution. Compute $\mathbb{E}(X_0)$ and $r_X(k)$ for $k\in\mathbf{N}$.

   (c) In this question, we assume $\alpha=0$ and $a_1>0$. Which kind of process is $(X_t)_{t\in\mathbf{Z}}$ and provide condition of the existence of a stationary causal second order solution. Compute $\mathbb{E}(X_0)$ and $r_X(k)$ for $k\in\mathbf{N}$.

   (d) Now and until the end, we study the general case $(\alpha,a_0,a_1)\in]=\mathbf{R}\times(0,\infty)\times[0,\infty)$. Prove that $(X_t)_{t\in\mathbf{Z}}$ is an affine causal process. Prove that the function $x\to\sqrt{1+x^2}$ is Lipshitzian and deduce that a sufficient condition for $(X_t)_{t\in\mathbf{Z}}$ to be a causal stationary second order process is:

$$|\alpha|+\sqrt{a_1}<1. \tag{2}$$

(e) For $|\alpha| < 1$, prove that if $(X_t)_{t\in\mathbf{Z}}$ is a causal stationary second order process then $(\varepsilon_t)_{t\in\mathbf{Z}}$ defined by

$$\varepsilon_t = \xi_t \sqrt{a_0 + a_1\Big(\sum_{i=0}^{\infty} \alpha^i\, \varepsilon_{t-1-i}\Big)^2} \quad \text{for any } t \in \mathbf{Z}, \tag{3}$$

is a causal stationary second order process and a weak white noise. Show that if $(\varepsilon_t)_{t\in\mathbf{Z}}$ is a causal stationary second order process then

$$a_1 + \alpha^2 < 1. \tag{4}$$

Compare this condition with (2).

(f) In Doukhan *et al.* (2016), it was established that under (4), then $(\varepsilon_t)_{t\in\mathbf{Z}}$ is a causal stationary second order process. Extend this property to $(X_t)_{t\in\mathbf{Z}}$. Under (4), compute $\mathbb{E}(X_0)$ and $r_X(k)$ for $k \in \mathbf{N}$.

(g) Deduce also $\mathbb{E}(X_t \mid (X_{t-s})_{s\in\mathbf{N}^*})$ and $\mathrm{var}(X_t \mid (X_{t-s})_{s\in\mathbf{N}^*})$. Is $(X_t)_{t\in\mathbf{Z}}$ a conditionaly heteroskedastic process?

(h) Assume now that $(X_1, \cdots, X_N)$ is observed and let $\theta = {}^t(\alpha, a_0, a_1)$. Provide the expression of the quasi-maximum likelihood estimator $\widehat{\theta}$ of $\theta$. Is $\widehat{\theta}$ a consistent estimator? What is its convergence rate?

(i) Provide forecasting of $X_{N+1}$ and $X_{N+1}^2$.

*Proof.* (a) If $\alpha = a_1 = 0$, then $(X_t)$ is a white noise and a stationary process without additional condition. And $E(X_0) = 0$, $r_X(k) = a_0$ if $k = 0$, else 0.

(b) If $a_1 = 0$ and $\alpha \neq 0$, then $(X_t)$ is a causal AR[1] process if $|\alpha| < 1$, a noncausal AR[1] process if $|\alpha| > 1$ and a non stationary process if $|\alpha| = 1$. We have $\mathbb{E}(X_0) = 0$, and $r_X(k) = a_0\, \alpha^k/(1-\alpha^2)$ when $|\alpha| < 1$, $r_X(k) = a_0\, \alpha^{2-k}/(\alpha^2 - 1)$ when $|\alpha| > 1$.

(c) If $a_1 \neq 0$ and $\alpha = 0$, then $(X_t)$ is a causal ARCH[1] process when $a_1 < 1$. We have $\mathbb{E}(X_0) = 0$, and $r_X(k) = a_0/(1-a_1)$ if $k = 0$, else 0.

(d) We have $X_t = F((X_s)_{s<t}) + \xi_t\, M((X_s)_{s<t})$, with $F((X_s)_{s<t}) = \alpha\, X_{t-1}$ and $M((X_s)_{s<t}) = \sqrt{a_0 + a_1\, X_{t-1}^2}$: $(X_t)$ is an affine causal process.
From finite increments theorem, we have $|\sqrt{1+x^2} - \sqrt{1+y^2}| \leq \sup_{z\in\mathbf{R}} |z|(1+z^2)^{-1/2}\, |x-y| \leq |x-y|$ for any $(x,y) \in \mathbf{R}^2$.
Therefore, with $M((X_s)_{s<t}) = \sqrt{a_0}\sqrt{1 + (\sqrt{a_1}\, X_{t-1}/\sqrt{a_0})^2}$, the Lipshitzian coefficients of $M$ are $\alpha_i(M) = \sqrt{a_0} \times \sqrt{a_1}/\sqrt{a_0} = \sqrt{a_1}$ for $i = 1$, and 0 else, while for $F$, $\alpha_i(F) = |\alpha|$ for $i = 1$, and 0 else. Therefore a sufficient condition of stationarity of a secon order solution is $\sum_i \alpha_i(F) + \sum_i \alpha_i(M) < 1$ implying $|\alpha| + \sqrt{a_1} < 1$.

(e) As $\varepsilon_t = X_t - \alpha\, X_{t-1}$, a finite linear combination of a stationary process, then $(\varepsilon_t)$ is a stationary process. Moreover, as $|\alpha| < 1$, we have $X_t = \sum_{i=0}^{\infty} \alpha^i \varepsilon_{t-i}$ and therefore we obtain (3). As $(X_t)$ is a causal process, this such a case for $(\varepsilon_t)$ and as it was done mnay times, since $\varepsilon_t = \xi_t G((\xi_s)_{s<t})$ we deduce $\mathbb{E}(\varepsilon_0) = 0$ and $r_\varepsilon(k) = Cte$ for $k = 0$ and 0 else: a weak white noise.
Moreover, since $\mathrm{var}(\varepsilon_t) = \mathbb{E}(\varepsilon_t^2) = \mathbb{E}(\varepsilon_0^2)$ for any $t$, we deduce, $\mathbb{E}(\varepsilon_t^2) = \mathbb{E}(\xi_t)\Big(a_0 + a_1 \mathbb{E}\big((\sum_{i=0}^{\infty} \alpha^i \varepsilon_{t-1-i})^2\big)\Big)$ from independence, implying $\mathbb{E}(\varepsilon_0^2) = \Big(a_0 + a_1\, \mathbb{E}(\varepsilon_0^2) \sum_{i=0}^{\infty} \alpha^{2i}\Big)$ and therefore $(1 - \alpha^2 - a_1)\mathbb{E}(\varepsilon_0^2) = a_0(1-\alpha^2)$. This is possible only if $1 - \alpha^2 - a_1 >$, *i.e.* condition (4).
We have $(\sqrt{a_1} + |\alpha|)^2 = a_1 + \alpha^2 + 2a_1|\alpha| \geq a_1 + \alpha^2$. Therefore if $a_1 + \alpha^2 < 1$ then $\sqrt{a_1} + |\alpha| < 1$.

(f) If $(\varepsilon_t)$ is a causal stationary second order process, then since $X_t = \sum_{i=0}^{\infty} \alpha^i \varepsilon_{t-i}$ and $\sum |\alpha^i| < \infty$, then $(X_t)$ is also a causal stationary second order process.
We have $\mathbb{E}(X_t) = 0$ and $r_X(k) = \frac{a_0}{1-a_1}\, \alpha^k/(1-\alpha^2)$.

(g) It is clear that $\mathbb{E}(X_t \mid (X_{t-s})_{s\in\mathbf{N}^*}) = F((X_{t-s})_{s\in\mathbf{N}^*}) = \alpha\, X_{t-1}$ and $\mathrm{var}(X_t \mid (X_{t-s})_{s\in\mathbf{N}^*}) = M^2((X_{t-s})_{s\in\mathbf{N}^*}) = a_0 + a_1\, X_{t-1}^2$. Therefore $(X_t)$ is a conditionally heteroskedastic process.

(h) After usual computations, we have

$$\widehat{\theta} = \operatorname*{argmin}_{\theta\in\Theta} \Big\{ \log(a_0) + \frac{X_1^2}{a_0} + \sum_{i=2}^{N} \log\big(a_0 + a_1 X_{i-1}^2\big) + \frac{(X_i - \alpha\, X_{i-1})^2}{a_0 + a_1 X_{i-1}^2} \Big\}$$

Since the number of Lipshitzian coefficients is finite, since the identification assumption is satisfied, then under (4), the estimator is strongly consistent.
Moreover, since the numbers of Lipzitzian coefficients of derivatives and second derivatives are also finite, then if $\mathbb{E}(\xi_0^4) < \infty$ and under (4), the estimator is strongly consistent, then $\widehat{\theta}$ satisfies a central limit theorem with a convergence rate $\sqrt{n}$.

(i) We have $\widehat{X}_{N+1} = \mathbb{E}(X_{N+1} \mid X_n, \ldots) = \alpha\, X_N$. Since $\alpha$ is unknown, it could be replaced by $\widehat{\alpha}$.
We have $\widehat{X}_{N+1}^2 = \mathbb{E}(X_{N+1}^2 \mid X_N, \ldots) = \alpha^2 X_N^2 + (a_0 + a_1 X_N^2)$. Since the parameters are unknown, they could be replaced by $\widehat{\alpha}$, $\widehat{a}_0$ and $\widehat{a}_1$.

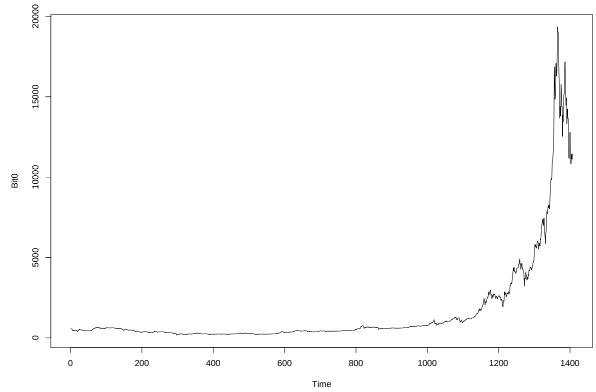
□

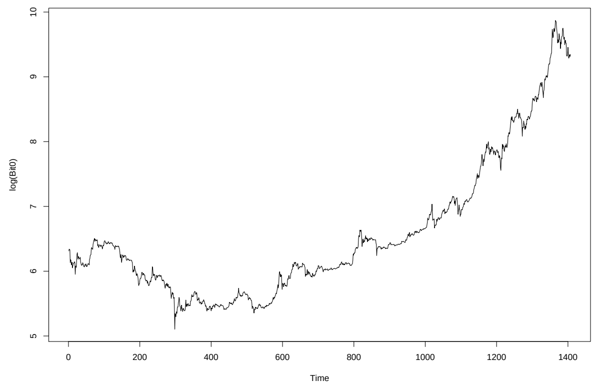
3. **Numerical part:** We study with `R` software the open daily historical data of Bitcoin from January 28 2014 to January 28 2018.

(a) First the following commands have been executed with figures exhibited below:

```
Bit=read.csv("C:/Users/Admin/Dropbox/Enseignement/M2 MO/TP/BTC-USD.csv")
Bit0=Bit$Open; n=length(Bit0)
plot.ts(Bit0); plot.ts(log(Bit0))
Y=log(Bit0); X1=c(1:n); X2=X1^2
Y.lm=lm(Y~X1+X2); summary(Y.lm)
```

Command `lm` realizes a least squares linear regression. Here there are the graphs and main numerical results:



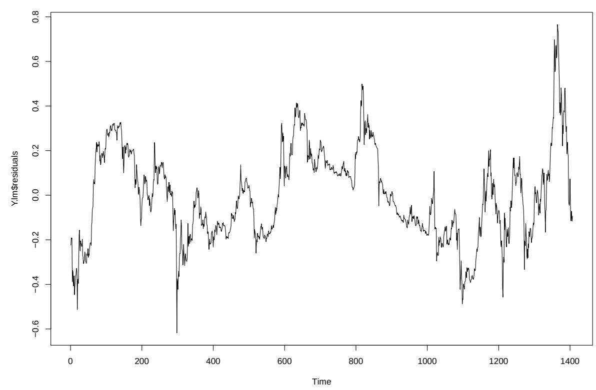

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  6.541e+00  1.645e-02  397.61   <2e-16 ***
X1          -4.034e-03  5.396e-05  -74.76   <2e-16 ***
X2           4.332e-06  3.711e-08  116.73   <2e-16 ***

Residual standard error: 0.2054 on 1404 degrees of freedom
Multiple R-squared:  0.9635,Adjusted R-squared:  0.9635
F-statistic: 1.854e+04 on 2 and 1404 DF,  p-value: < 2.2e-16
```

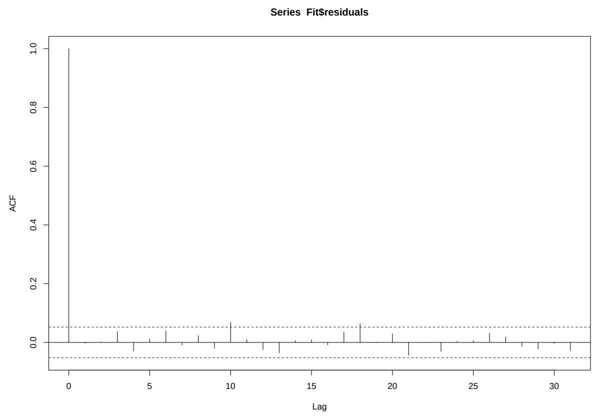
*Question II.1: Explain what is done.*

(b) New commands are then executed:

```
plot.ts(Y.lm$residuals)
Fit=arima(Y.lm$residuals, order = c(1,0,2))
acf(Fit$residuals)
Box.test(Fit$residuals, lag = 20,"Ljung-Box", fitdf=3)
```

Here there are graphs and numerical results:




```
Box-Ljung test
data:  Fit$residuals
X-squared = 25.557, df = 17, p-value = 0.08292
```

*Question II.2: Explain what is done (notably why we use* `fitdf=3`*) and explain which conclusions you deduce.*

(c) The following sequence of commands is then executed:

```
pred=predict(Fit,n.ahead=1); pred[1]
exp(pred$pred[1]+sum(Y.lm$coeff*c(1,(n+1),(n+1)^2)))
```

The results are the following:

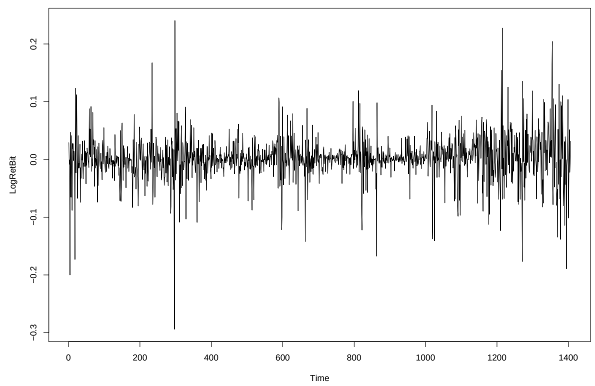
```
>-0.09092553
> 11582.19
```

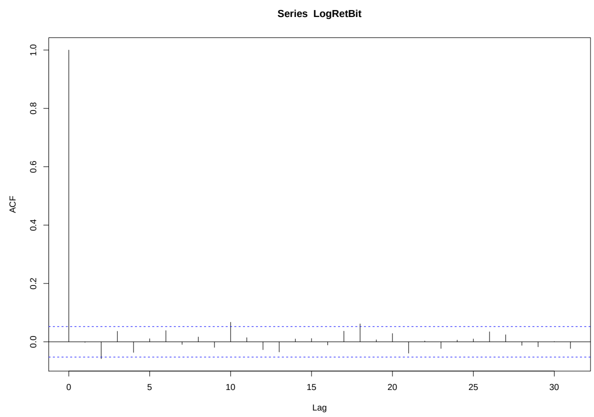
*Question II.3: What is done here and what are your conclusions?*

(d) Finally, the following sequence of commands is executed:

```
LogRetBit=log(Bit0[2:n]/Bit0[1:(n-1)])
plot.ts(LogRetBit); acf(LogRetBit)
library(fGarch)
FitLogRet1=garchFit(~garch(1,1),data=LogRetBit,trace=FALSE)
summary(FitLogRet1)
FitLogRet2=garchFit(~garch(1,2),data=LogRetBit,trace=FALSE)
summary(FitLogRet2)
```

The graphs and results are the following:




```
> Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu     1.646e-03   7.012e-04    2.347 0.018911 *
omega  2.894e-05   7.565e-06    3.825 0.000131 ***
alpha1 1.690e-01   2.155e-02    7.845 4.44e-15 ***
beta1  8.365e-01   1.774e-02   47.139  < 2e-16 ***

Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  2253.533  0
 Shapiro-Wilk Test  R    W      0.9142998 0
 Ljung-Box Test     R    Q(10)  31.40408  0.0005030119
 Ljung-Box Test     R    Q(15)  34.67742  0.002732705
 Ljung-Box Test     R    Q(20)  42.10945  0.002676158
 Ljung-Box Test     R^2  Q(10)  7.152754  0.7109496
 Ljung-Box Test     R^2  Q(15)  11.73218  0.6991771
 Ljung-Box Test     R^2  Q(20)  16.21253  0.7033551
 LM Arch Test       R    TR^2   9.629902  0.648393

Information Criterion Statistics:
      AIC       BIC       SIC      HQIC
-3.971264 -3.956332 -3.971280 -3.965683

> Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu     1.711e-03   7.038e-04    2.431 0.015041 *
omega  3.529e-05   1.068e-05    3.305 0.000949 ***
alpha1 2.098e-01   3.277e-02    6.401 1.54e-10 ***
beta1  4.464e-01   1.945e-01    2.295 0.021745 *
beta2  3.485e-01   1.755e-01    1.985 0.047127 *

Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  2360.269  0
 Shapiro-Wilk Test  R    W      0.9149907 0
```

```
 Ljung-Box Test     R    Q(10)  30.51412  0.000705495
 Ljung-Box Test     R    Q(15)  33.53633  0.003954055
 Ljung-Box Test     R    Q(20)  40.94799  0.003782937
 Ljung-Box Test     R^2  Q(10)  5.451614  0.8590436
 Ljung-Box Test     R^2  Q(15)  8.699528  0.8926969
 Ljung-Box Test     R^2  Q(20)  13.33315  0.8626366
 LM Arch Test       R    TR^2   7.509703  0.8221771

Information Criterion Statistics:
      AIC       BIC       SIC      HQIC
-3.974214 -3.955549 -3.974239 -3.967238
```

*Question II.4: What is done here and which model could you chose?*