rattrapage 2012

Exo 1.

1. $$L(\lambda) = \prod_{i=1}^{n} C_k^{x_i} \left(\frac{\lambda}{k}\right)^{x_i} \left(1-\frac{\lambda}{k}\right)^{k-x_i}$$

$$= \prod_{i=1}^{n} C_k^{x_i} \left(\frac{\lambda}{k}\right)^{\sum x_i} \left(1-\frac{\lambda}{k}\right)^{nk-\sum x_i}$$

$$H(\lambda) = \ln L(\lambda) = \ln \prod_{i=1}^{n} C_k^{x_i} + \sum x_i (\ln\lambda - \ln k) + (nk - \sum x_i) \ln\left(1-\frac{\lambda}{k}\right)$$

$$H'(\lambda) = \frac{\sum x_i}{\lambda} + \frac{nk-\sum x_i}{1-\frac{\lambda}{k}} \times \left(-\frac{1}{k}\right) = \frac{\sum x_i}{\lambda} - \frac{nk-\sum x_i}{k-\lambda} = 0$$

$$k\sum x_i - \cancel{\lambda \sum x_i} = \lambda n k - \cancel{\lambda \sum x_i}$$

$$\left.\begin{array}{l} \lambda = \frac{1}{n}\sum x_i \\ H''(\lambda) = -\frac{\sum x_i}{\lambda^2} - \frac{nk-\sum x_i}{(k-\lambda)^2} < 0 \end{array}\right\} \Rightarrow \hat{\lambda}_{MV} = \frac{1}{n}\sum x_i$$

2. $$\mathbb{E}(\hat{\lambda}_{MV}) = \mathbb{E}X_i = \lambda, \quad Var(\hat{\lambda}_{MV}) = \frac{Var(X_i)}{n} = \frac{\lambda}{n}\left(1-\frac{\lambda}{k}\right).$$

$$\frac{\lambda}{k}\left(1-\frac{\lambda}{k}\right) \le \frac{1}{4} \Rightarrow \frac{\lambda}{n}\left(1-\frac{\lambda}{k}\right) = \frac{k}{n}\frac{\lambda}{k}\left(1-\frac{\lambda}{k}\right) \le \frac{k}{4n}$$

$$\hat{\lambda}_{MV} \sim \mathcal{N}\left(\lambda, \frac{k}{4n}\right)$$

$$\frac{\hat{\lambda}_{MV}-\lambda}{\frac{1}{2}\sqrt{\frac{k}{n}}} \sim \mathcal{N}(0,1)$$

Supposons $\mathbb{P}(|Y| \le q_\alpha) = \alpha$.

$$\mathbb{P}\left(\frac{|\hat{\lambda}_{MV}-\lambda|}{\frac{1}{2}\sqrt{\frac{k}{n}}} \le q_\alpha\right) = \alpha$$

$$IC_\alpha = \left[\hat{\lambda}_{MV} - \frac{q_\alpha}{2}\sqrt{\frac{k}{n}}, \quad \hat{\lambda}_M + \frac{q_\alpha}{2}\sqrt{\frac{k}{n}}\right]$$

3. $k=25,\ n=100,\ \hat{\lambda}_{MV}=5{,}5$.

a) $X \sim Bin\left(25, \frac{\lambda}{25}\right)$

b) $$\hat{T} = \frac{|\hat{\lambda}_n - 6|}{\frac{1}{2}\frac{5}{10}} = \frac{0.5}{\frac{1}{4}} = 2$$

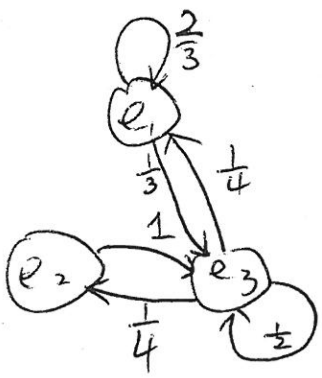
$\mathbb{P}(|Y| > 1{,}96) = 0{,}05$

$W = \{\hat{T} > 1{,}96\}$, $\hat{T} \in W \Rightarrow$ On rejette $H_0 : \lambda = 6$

c) $$\hat{T} = \frac{|6{,}35 - 6|}{\frac{1}{2}\frac{5}{\sqrt{n}}} = \frac{0{,}7\sqrt{n}}{5}$$

$$\hat{T} \notin W \Rightarrow \frac{0{,}7\sqrt{n}}{5} \leq 1{,}96 \Rightarrow \sqrt{n} \leq \frac{1{,}96 \times 5}{0{,}7} \Rightarrow n \leq 14^2 = 196$$

Exo 2.



1. $$M = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 1 & \frac{1}{2} \end{pmatrix}$$. Oui

2. $$\begin{cases} \frac{2}{3}u_1 + \frac{1}{4}u_3 = u_1 \\ \frac{1}{4}u_3 = u_2 \\ \frac{1}{3}u_1 + u_2 + \frac{1}{2}u_3 = u_3 \\ u_1 + u_2 + u_3 = 1 \end{cases} \Rightarrow \begin{cases} u_1 = \frac{3}{8} \\ u_2 = \frac{1}{8} \\ u_3 = \frac{1}{2} \end{cases}$$

3. ① $e_3 \xrightarrow{\frac{1}{4}} e_1 \xrightarrow{\frac{1}{3}} e_3$

② $M^2 = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{19}{36} & \frac{1}{4} & \frac{7}{24} \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{8} \\ \frac{7}{18} & \frac{1}{2} & \frac{7}{12} \end{pmatrix}$

$P(X_3 = e_1 \mid X_1 = e_1) = M^2_{(1,1)} = \frac{19}{36}$

4. $u_3 > u_1,\ u_3 > u_2.$ $\quad e_3$

Exo 3.

1. $\int_\theta^\infty f_\theta(x)dx = \int_\theta^\infty \frac{c}{x^3}dx = \left[-\frac{cx^{-2}}{2}\right]_\theta^\infty = \frac{c}{2\theta^2} = 1 \Rightarrow c = 2\theta^2$

2. $F_X(t) = \int_\theta^t \frac{2\theta^2}{x^3}dx = \left[-\frac{\theta^2}{x^2}\right]_\theta^t = 1 - \frac{\theta^2}{t^2}$

$Y = \min(X_1, X_2)$

$$\begin{aligned} F_Y(t) = \mathbb{P}(\min(X_1,X_2) \leq t) &= 1 - \mathbb{P}(\min(X_1,X_2) > t) \\ &= 1 - \mathbb{P}(X_1 > t)\mathbb{P}(X_2 > t) \\ &= 1 - (1 - F_X(t))^2 \\ &= 1 - \frac{\theta^4}{t^4} \end{aligned}$$

$g_\theta(t) = F_Y'(t) = \frac{4\theta^4}{t^5}, \quad t \in [\theta, \infty)$

3. $\mathbb{E}(X) = \int_\theta^\infty x f_\theta(x)dx = \int_\theta^\infty \frac{2\theta^2}{x^2}dx = \left[-\frac{2\theta^2}{x}\right]_\theta^\infty = 2\theta$

$\mathbb{E}(\hat{\theta}_1) = \frac{1}{4} \times 2 \times \mathbb{E}(X) = \theta.$

$$\mathbb{E}(Y) = \int_\theta^\infty x\, g_\theta(x)\,dx = \int_\theta^\infty \frac{4\theta^4}{x^4}\,dx = \left[-\frac{4\theta^4}{3x^3}\right]_\theta^\infty = \frac{4\theta}{3}$$

$$\mathbb{E}(\hat{\theta}_2) = \frac{3}{4}\mathbb{E}(Y) = \theta$$

4. $$\mathbb{E}(X^2) = \int_\theta^\infty \frac{2\theta^2}{x}\,dx = \left[2\theta^2 \ln x\right]_\theta^\infty = \infty \Rightarrow Var(X) = \infty$$

$$RQM(\hat{\theta}_1) = Var(\hat{\theta}_1) = \frac{1}{16} \times 2 \times Var(X) = \infty$$

$$\mathbb{E}(Y^2) = \int_\theta^\infty \frac{4\theta^4}{x^3}\,dx = \left[-\frac{2\theta^4}{x^2}\right]_\theta^\infty = 2\theta^2$$

$$Var(Y) = 2\theta^2 - \left(\frac{4\theta}{3}\right)^2 = \frac{2}{9}\theta^2$$

$$RQM(\hat{\theta}_2) = Var(\hat{\theta}_2) = \frac{9}{16} Var\, Y = \frac{\theta^2}{8}$$

5. $RQM(\hat{\theta}_2) < RQM(\hat{\theta}_1) \Rightarrow \hat{\theta}_2$ est meilleur