Scale - Invariant Nature of Stock Market to Detect Speculative Bubble

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1. **Short History**

2. **Speculative Bubble**

3. **Scale - Invariant**

4. **Calibration**

5. **Exemples**

6. **References**
**Tulip Mania**

**Figure:** One of the first speculative bubble, forward contract tulip bulbs, 1634-1637.
Non Exhaustive List

- 1929, Wall Street crash (margin buying)
- 1973, oil crisis (oil embargo)
- 1980, Latin American crisis (external debt)
- 1987, Black Monday (program trading, overvaluation, illiquidity)
- 1992, Black Wednesday (pound sterling)
- 1997, Asian crisis (Thai baht)
- 1998, Russian crisis (debt crisis)
- 2001, dot-com bubble (overvaluation)
- 2008, Subprimes crisis (subprime mortgage crisis)
- 2011, Sovereign debt crisis
**DEFINITION**

A bubble grows, grows, grows, but the rise is based on nothing but on thin air, so that, it is vulnerable to a sudden burst.
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**An other definition :**

The simple way to understand the formation of a speculative bubble is when the demand excess the supply, the price is auto-referential and no longer depend on the fair value. Of course, in that case, classical arbitrage between a class of assets fails.
No arbitrage?...

So, is detecting so obvious? We could think that if a couple of asset has been well cointegrate for years, then, as soon their relationship is broken for weeks, we are observing a bubble formation on one of the two asset. It could be true, just think to the guy who was pair trading with italian bond versus german bond before the Euro lunch.

But (there is always a but)! If cointegration is broken, it’s maybe just because the fundamentals of one of the two assets has changed, for exemple Apple ans its Iphone. Worst! Major US index, let say S&P 500 and DJIA, have very close motion, and they could be (perhaps?) cointegrate, but if one crash, the other will not wait very long before crashing too...
Speculative Bubble

**Not so easy to detect**

- Isaac Newton as the master of the Mint in London: “I can calculate the motion of heavenly bodies, but not the madness of people”, 1720.

- Alan Greenspan as the chairman of the FED: “As events evolved, we recognized that, despite our suspicions, it was very difficult to definitively identify a bubble until after the fact that is, when its bursting confirmed its existence”, 2002.
An other definition, cont.

In other words, agents exaggerate the extent of the improvement in economic fundamentals. Price formation is not rational, and unfortunately, imitation is rational.
**Dot-Com Bubble**

*Figure:* In black, Nasdaq 100, in red, S&P 500, 1986/11/26 - 2012/02/23
**Figure:** In black, Nasdaq 100, in gold, Eiffel Tower (warning, wrong scaling).
**Figure:** Power law fit of the DJIA before the crash of October 1987.
Scale - Invariance

**Definition**

Let $f$ an observable, then, $f$ is scale invariant if there exists a function $\mu$ such that for any change $x \rightarrow \lambda x$,

$$f(x) = \mu f(\lambda x)$$

General solution :

$$f(x) \propto x^\alpha$$
Figure: Periodic law fit of the DJIA before the crash of October 1987. The graph shows an apparent scale-invariance.
Log - Periodic Power Law

**Model**

More general solution

\[ f(x) = x^\alpha \Pi(\ln(x)/\ln(\lambda)) \]

If \( f \), our observable, is the bubble formation, we clearly have a scale-invariant process. Due to oscillation we would like to have \( \Pi \) as a periodic function, hence, a reasonable solution is the first term of its Fourier expansion,

\[ \Pi(\ln(x)/\ln(\lambda)) = A + B \cos\left(\frac{\omega'}{2\pi} \ln(x) + \Phi\right) \]

with \( x \) the time close to market crash \( t_c \), the critical event,

\[ x = (t_c - t) \]
**Model**

During bubble formation, we obtain the model,

\[ SB(S_t) \approx a_1 + a_2(t_c - t)^\alpha + b(t_c - t)^\alpha \cos\left(\frac{\omega'}{2\pi} \ln(t_c - t) + \Phi\right) \]

With \( SB \) the speculative bubble, \( S_t \) the price of the financial instrument, \( t_c \) the time where the crash occurs.
Log - Periodic Power Law

Model

During bubble formation, we obtain the model,

\[ SB(S_t) \approx a_1 + a_2(t_c - t)^\alpha + b(t_c - t)^\alpha \cos(\frac{\omega'}{2\pi} \ln(t_c - t) + \Phi) \]

With \( SB \) the speculative bubble, \( t_c \) the time where the crash occurs. The first term is the intercept.
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With \( SB \) the speculative bubble, \( t_c \) the time where the crash occurs. The first term is the intercept, the second is a pure power law describing the exponential growth of the price close to the critical event.
During bubble formation, we obtain the model,

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With $SB$ the speculative bubble, $t_c$ the time where the crash occurs. The first term is the intercept, the second is a pure power law describing the exponential growth of the price close to the critical event and the last term describes the particular oscillation and contraction of the bubble.
Loss function

**Residual Sum of Square**

\[ RSS(\Theta) = \sum_{t=1}^{T} (\ln(S_t) - a_1 - a_2(t_c - t)^\alpha - b(t_c - t)^\alpha \cos(\omega \ln(t_c - t) + \Phi))^2 \]

\[ \Theta = (a_1, a_2, b, t_c, \alpha, \omega, \Phi) \]
3-linear, 4 non-linear

\[ \text{RSS}(\Theta) = \sum_{t=1}^{T} \left( \ln(S_t) - a_1 - a_2(t_c - t)\alpha - b(t_c - t)^\alpha \cos(\omega \ln(t_c - t) + \Phi) \right)^2 \]

\[ \equiv \sum_{t=1}^{T} \left( \ln(S_t) - a_1 - a_2 f_t - bg_t \right)^2 \]

2-step procedure

\[ \{ a_1^*, a_2^*, b^* \} = \text{argmin}_{a_1, a_2, b} \text{RSS}(a_1, a_2, b, t_c, \alpha, \omega, \Phi) \]

\[ \min_{\Theta} \text{RSS}(\Theta) = \min_{t_c, \alpha, \omega, \Phi} \{ \text{RSS}(a_1^*, a_2^*, b^*, t_c, \alpha, \omega, \Phi) \} \]
3-LINEAR, 4 NON-LINEAR

\[
RSS(\Theta) = \sum_{t=1}^{T} \left( \ln(S_t) - a_1 - a_2(t_c - t)^{\alpha} - b(t_c - t)^{\alpha} \cos(\omega \ln(t_c - t) + \Phi) \right)^2
\]

\[
\equiv \sum_{t=1}^{T} \left( \ln(S_t) - a_1 - a_2f_t - bg_t \right)^2
\]

2-STEP PROCEDURE

\[
\{a_1^*, a_2^*, b^*\} = \arg\min_{a_1, a_2, b} RSS(a_1, a_2, b, t_c, \alpha, \omega, \Phi)
\]

\[
\min_{\Theta} RSS(\Theta) = \min_{t_c, \alpha, \omega, \Phi} \{RSS(a_1^*, a_2^*, b^*, t_c, \alpha, \omega, \Phi)\}
\]

The first order condition is, with \(y_t = \ln(S_t)\),

\[
\begin{pmatrix}
\sum_{t=1}^{T} f_t \\
\sum_{t=1}^{T} f_t^2 \\
\sum_{t=1}^{T} f_t g_t
\end{pmatrix}
\cdot
\begin{pmatrix}
\sum_{t=1}^{T} g_t \\
\sum_{t=1}^{T} f_t g_t \\
\sum_{t=1}^{T} g_t^2
\end{pmatrix}
\cdot
\begin{pmatrix}
a_1^* \\
a_2^* \\
b^*
\end{pmatrix}
= \begin{pmatrix}
\sum_{t=1}^{T} y_t \\
\sum_{t=1}^{T} y_t f_t \\
\sum_{t=1}^{T} y_t g_t
\end{pmatrix}
\]
\[ SB(S_t) \approx a_1 + a_2(t_c - t)^\alpha + b(t_c - t)^\alpha \cos(\omega \ln(t_c - t) + \Phi) \]

**Reduce the number of non-linear parameters**

Remember that \( \cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y) \), hence,

\[ SB(S_t) \approx a_1 + a_2(t_c - t)^\alpha + a_3(t_c - t)^\alpha \cos(\omega \ln(t_c - t)) + a_4(t_c - t)^\alpha \sin(\omega \ln(t_c - t)) \]

\[ \equiv a_1 + a_2f_t + a_3g_t + a_4h_t \]

\( \Theta = (a_1, a_2, a_3, a_4, t_c, \alpha, \omega), \) 4 linear parameters, 3 non-linear.

\[
\begin{pmatrix}
T \\
\sum f_t \\
\sum g_t \\
\sum h_t
\end{pmatrix}
\begin{pmatrix}
\sum f_t \\
\sum f_t^2 \\
\sum f_tg_t \\
\sum f_th_t
\end{pmatrix}
\begin{pmatrix}
\sum g_t \\
\sum f_tg_t \\
\sum g_t^2 \\
\sum g_th_t
\end{pmatrix}
\begin{pmatrix}
\sum h_t \\
\sum f_th_t \\
\sum g_th_t \\
\sum h_t^2
\end{pmatrix}
\begin{pmatrix}
a_1^* \\
a_2^* \\
a_3^* \\
a_4^*
\end{pmatrix}
= 
\begin{pmatrix}
\sum y_t \\
\sum y_tf_t \\
\sum y_tg_t \\
\sum y_th_t
\end{pmatrix}
NON LINEAR PARAMETERS

- No closed formula for \( \{t_c, \alpha, \omega\} \).
- We can use a matheuristic algorithm like genetic algorithm or taboo search, and for each value of \( \{t_c, \alpha, \omega\} \) with the corresponding value \( \{a_1, a_2, a_3, a_4\} \).
- No theoretical justification to obtain the global minimum.
- We need to check the robustness of the algorithm!
Robustness

- Main interest for $t_c$, the time where the crash occurs.
  Let us note $[t_1, t_2]$ the training period, with $t_2 < t_c$, algorithm be very sensitive to the choice of window.

  Hence, we compute the cost function for different $t_1$, $t_2$ and $t_c$ to check if the map admits several local minimum at very distant time, just one global.
**Robustness - DJIA**

**Figure:** \( t_1 \) min = 1985-09-20, \( t_1 \) max = 1986-07-09, \( t_c \) min = 1987-08-04, \( t_c \) max = 1989-12-23, \( t_2 \) = 1987-08-03.
**Figure:** Cost function for 41 different value of $t_1$ and 50 of $t_c$ varying between 1986-07-09 and 1987-08-03, the black line correspond to minimal value for $t_c$.  

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**Robustness - DJIA**
**ROBUSTNESS - DJIA**

**Figure:** Cost function for 50 different value of $t_c$ and 41 of $t_1$ varying between $t_2 + 1 = 1987-08-04$ and $t_2 + 100 = 1989-12-23$. 
**Robustness - DJIA**

**Figure:** 3D plot of cost function for different fixed value of $t_1$ and $t_c$. In red, the corresponding minima of $t_c$ with respect to different value of $t_1$. 
Figure: Two optimal values obtained corresponding to 28-08 and 09-01 1987.
**Figure:** $t_2 \text{ min} = 11-28-1986$, $t_2 \text{ max} = 02-05-1988$, $t_c \text{ min} = 12-01-1986$, $t_c \text{ max} = 02-08-1988$, $t_1 = 09-20-1985$. 
**Figure:** 3D plot of cost function for different fixed value of $t_2$ and $t_c$. In red, the corresponding minima of $t_c$ with respect to different value of $t_2$. 
**Figure**: Fitted log-periodic power law, optimal values between 08-21-1987 to 10-07-87 and 12-02-87 to 01-07-88 (dashed lines).
**Light Crude**

**Figure:** Fitted log-periodic power law before the crash of 07-2008.
**Figure:** Fitted log-periodic power law before the crash of 05-2011.
BOMBAY SENSEX INDEX STOCK

**Figure:** Fitted log-periodic power law before the crash of 05-2006.
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